Young tableaux and the n-dimensional 2-complex

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1 The 2-Complex

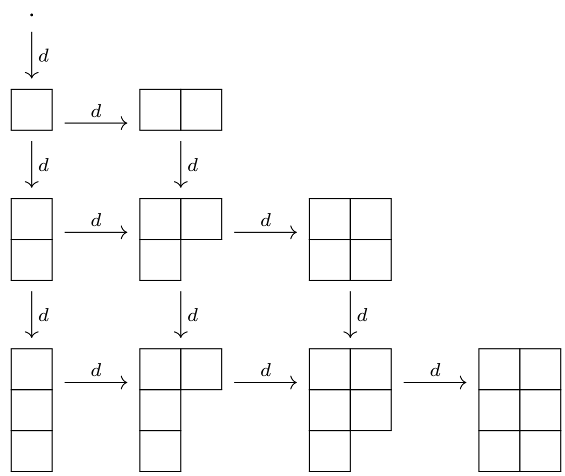
In [2] it was observed that certain function spaces on $\mathbb{R}^3$ are connected by differential operators, such that this diagram commutes, and three consecutive operators vanish:

$$
\begin{array}{ccccccc}
H^1 & \xrightarrow{\text{grad}} & H(\text{curl}) & \xrightarrow{\text{curl}} & H(\text{div}) & \xrightarrow{\text{div}} & L^2 \\
\downarrow{\scriptstyle \text{grad}} & & \downarrow{\scriptstyle \text{def}} & & \downarrow{\scriptstyle \text{dev grad}^T} & & \downarrow{\scriptstyle \text{grad}} \\
H(\text{curl}) & \xrightarrow{\text{def}} & H_{cc}(\mathbb{S}) & \xrightarrow{\text{curl}} & H_{cd}(\mathbb{T}) & \xrightarrow{\text{div}} & H^{-1}(\text{curl}) \\
\downarrow{\scriptstyle \text{curl}} & & \downarrow{\scriptstyle \text{curl}^T} & & \downarrow{\scriptstyle \text{sym curl}^T} & & \downarrow{\scriptstyle \text{curl}} \\
H(\text{div}) & \xrightarrow{\text{dev grad}} & H_{dc}(\mathbb{T}) & \xrightarrow{\text{sym curl}} & H_{dd}(\mathbb{S}) & \xrightarrow{\text{div}} & H^{-1}(\text{div}) \\
\downarrow{\scriptstyle \text{div}} & & \downarrow{\scriptstyle \text{div}} & & \downarrow{\scriptstyle \text{div}} & & \downarrow{\scriptstyle \text{div}} \\
L^2 & \xrightarrow{\text{grad}} & H^{-1}(\text{curl}) & \xrightarrow{\text{curl}} & H^{-1}(\text{div}) & \xrightarrow{\text{div}} & H^{-1}
\end{array}
$$

with $\operatorname{def} u = \frac{1}{2}(\nabla u + \nabla u^T)$, $\operatorname{dev}\eta = \eta - \frac{1}{3}\operatorname{tr}\eta I$, $\operatorname{curl}^T \gamma = (\operatorname{curl}\gamma^T)^T$

1.1 2-Complex of Young tableaux

Spaces of differential forms with certain symmetries are represented by Young tableaux, a collection of boxes with numbers in the boxes. The number of boxes corresponds to the number of arguments of the form. The three-dimensional 2-complex above is symmetric with respect to the main diagonal, so it's enough to look only at the lower-left triangle. The left-most column of the diagram corresponds to the usual de Rham complex: The fully anti-symmetric forms are represented by vertically stacked boxes. All tableaux have at most two columns which means that the spaces are sub-spaces of the tensor product of two anti-symmetric forms. A vertical d-operator adds a box in the first column, a horizontal d-operator adds a box in the second column. The general algebraic framework in the context of Young tableaux goes back to [1].



All spaces are irreducble sub-spaces of tensor products of two differential forms, so called double forms, see [3] on double-forms.

2 Introduction

Let V be an n-dimensional vector spaces over $\mathbb{R}$ with basis elements $e_1, \ldots, e_n$, and e^i are basis elements for the dual space V'.

Let $\mathrm{Lin}^k(V)$ be the space of k-linear forms on V. Its vector-space dimension is n^k.

Let $GL(V)$ be the group of invertible linear maps $g : V \to V$. For $v = v_1 \otimes \ldots \otimes v_k$, we define $g(v) = g(v_1) \otimes \ldots g(v_k)$ as diagonal action.

Consider 2-linear forms $A \in \mathrm{Lin}^2(V)$. They can be split into symmetric forms

$$A^{sym}(v_1, v_2) := \tfrac{1}{2}\left\{A(v_1, v_2) + A(v_2, v_1)\right\}$$

and skew-symmetric forms

$$A^{skw}(v_1, v_2) := \tfrac{1}{2}\left\{A(v_1, v_2) - A(v_2, v_1)\right\},$$

such that $A(v_1, v_2) = A^{sym}(v_1, v_2) + A^{skw}(v_1, v_2)$. Taking symmetric and skew-symmetric parts are projections: taking the symmetric part of the symmetric part keeps the space unchanged. This splitting is invariant with respect to the diagonal group action, i.e.

$$\widetilde{A}(v_1, v_2) = A^{sym}(g(v_1), g(v_2))$$

is a symmetric 2-form, and similar for skew-symmetric forms. This is since the projections to sub-spaces are built from permuting arguments, and the permutations σ commute with the diagonal group action g.

The vector-space dimension of 2-forms is n^2, the dimension of symmetric forms is $\frac{n(n+1)}{2}$, and of skew-symmetric is $\frac{n(n-1)}{2}$.

A k-form can be identified with its coefficient tensor

$$a_{i_1,\ldots i_k} = A(e_{i_1}, \ldots, e_{i_k})$$

The symmetric and skew-symmetric forms correspond to symmetric and skew-symmetric coefficient matrices.

We want to see how to split higher order forms into invariant sub-spaces. We can fully symmetrize a form, e.g. for a 3-form we get the fully symmetric forms as

$$A^{sym}(v_1, v_2, v_3) = \frac{1}{|S_3|} \sum_{\sigma \in S_3} A(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}),$$

and the fully skew-symmetric forms

$$A^{skw}(v_1, v_2, v_3) = \frac{1}{|S_3|} \sum_{\sigma \in S_3} \operatorname{sign}(\sigma) A(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}).$$

For example, the space of 3-linear forms in $\mathbb{R}^3$ has dimension 27. We will see, the fully symmetric forms have dimension 10. By symmetry, the form is defined by specifying the the elements $a_{111}, a_{112}, a_{113}, a_{122}, a_{123}, a_{133}, a_{222}, a_{223}, a_{233}, a_{333}$ of the coefficient tensor. The fully skew-symmetric forms are of dimension 1, and a_{123} is a represents for these forms. We will see that there are two more invariant sub-spaces, each one of dimension 8. So we get $10 + 2 \cdot 8 + 1 = 27$. The first one is formed by a combination of partial symmetrizing

$$aA(v_1, v_2, v_3) = \frac{1}{2}\left\{A(v_1, v_2, v_3) + A(v_2, v_1, v_3)\right\}$$

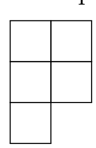
followed by a partial anti-symmetrizing

$$baA(v_1, v_2, v_3) = \frac{1}{2}\left\{aA(v_1, v_2, v_3) - aA(v_3, v_2, v_1)\right\}$$

The second one is obtained by symmetrizing with respect to v_1 and v_3, and then anti-symmetrizing for v_1 and v_2. The operators a and b are examples of row- and column stabilizers in combinatorics, see below.

3 Young tableaux

In the context of Young tableau, a shape $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_r$ non-negative integers is a collection of top-left aligned boxes like



,

where λ_j gives the number of boxes in the j^{th} row. $|\lambda| = \sum \lambda_i$ is called the size of the shape, we write $\lambda \vdash k$ for $k = |\lambda|$.

A standard Young tableau (SYT) is a shape filled with distinct natural numbers from $\{1, \ldots, |\lambda|\}$ which are row-wise and column-wise strictly increasing. All standard Young tableaux of shape $(2, 2, 1)$ are

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}.$$

We write

$$\mathrm{SYT}(\lambda)$$

for the set of all standard Young tableaux of shape λ. The number of tableaux of shape λ is given by the hook-length formula

$$\frac{|\lambda|!}{\prod_{\mathrm{boxes}\square} h(\square)},$$

where the hook-length $h(\square)$ of a box $\square$ is 1 plus number of boxes to the right plus number of boxes below. For the $(2,2,1)$ shape, the hook lengthes for each box are

$$\begin{array}{|c|c|}\hline 4&2\\\hline 3&1\\\hline 1\\\cline{1-1}\end{array},$$

and thus

$$\frac{5!}{4\cdot 2\cdot 3\cdot 1\cdot 1} = 5$$

confirms that there exist 5 Young tableaux to that shape.

A semi-standard Young tableau (SSYT) with entries at most n is a shape filled with natural numbers from $\{1,\dots n\}$ which are row-wise non-decreasing, and column-wise increasing. All semi-standard Young tableaux of shape $(2,1)$ of dimension $n=3$ are

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}$$

The hook-content formula allows to compute the number of SSYT via

$$\prod_{boxes\square} \frac{n+c_\square}{h_\square},$$

with the content $c(\square) = i - j$, where i and j are column and row numbers of the box $\square$ within the tableaux. For the $(2,1)$ shape we have hook-length and content:

$$h = \begin{array}{|c|c|}\hline 3&1\\\hline 1\\\cline{1-1}\end{array} \qquad c = \begin{array}{|c|c|}\hline 0&1\\\hline -1\\\cline{1-1}\end{array}$$

Thus, we confirm there are

$$\frac{3+0}{3}\,\frac{3+1}{1}\,\frac{3-1}{1} = 8$$

SSYT for the shape $(2,1)$ and $\dim V = 3$

4 Projectors

Let T be a standard tableau of the shape λ of size k (in short: $T \in \mathrm{SYT}(\lambda)$ for $\lambda \vdash k$). Define the row stabilizer

$$R(T) = \{\sigma \in S_k : \sigma \text{ leaves the sets of row entries invariant}\}$$

and column stabilizer

$$C(T) = \{\sigma \in S_k : \sigma \text{ leaves the sets of column entries invariant}\}$$

$$\begin{aligned} a(T) &:= \sum_{\sigma \in R(T)} \sigma \\ b(T) &:= \sum_{\sigma \in C(T)} \mathrm{sgn}(\sigma)\sigma \end{aligned}$$

and the Young symmetrizer

$$c(T) = a(T) \cdot b(T)$$

which becomes the Young projector after rescaling

$$e(T) = \frac{1}{\prod_{\Box} h(\Box)} c(T)$$

Thus $b(T)$ antisymmetrizes along columns and $a(T)$ symmetrizes along rows (i.e. $c(T)v = a(T)(b(T)v)$). When we work with forms, which are dual to the tensor-spaces, we first apply a and then b.

The Young-projector of the SYT T applied to the k-form A gives

$$aA(v_1, \ldots v_k) = \sum_{\sigma \in R(T)} A(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

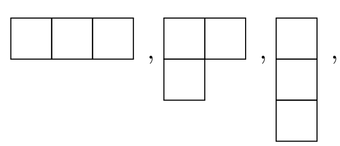
and

$$baA(v_1, \ldots v_k) = \sum_{\sigma \in C(T)} \mathrm{sign}(\sigma)\, aA(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

Schur-Weyl:

$$\{A : V^k \to \mathbb{R}\} = \sum_{\lambda \vdash k} \sum_{T \in SYT(\lambda)} b_T a_T A(\ldots)$$

Example:
$k = 3$, $n = 3$ All shapes of size 3 are:



and SYT are

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}.$$

The dimension of the sub-spaces is the number of SSYTs. These numbers are 10, 8, 8, 1, summing up to 27. For general $n \in \mathbb{N}$, the hook-content formula gives dimensions

$$\frac{n(n+1)(n+2)}{6}, 2 \cdot \frac{(n-1)n(n+1)}{3}, \frac{n(n-1)(n-2)}{6}$$

for all shapes of size 3, summing up to n^3.

5 Constructing a basis

Fix a SYT T of shape λ. For every $I \in \mathrm{SSYT}(\lambda)$ we define a basis representant

$$\widetilde{\varphi}_I(v_1, \ldots, v_{|\lambda|}) := \prod_{\square \in \lambda} e^{I_\square}(v_{T_\square}).$$

Then the actual basis is obtained by Young-symmetrization

$$\varphi_I := c(T)\widetilde{\varphi}_I.$$

Example: we choose the SYT

$$T = \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array},$$

and recall the 8 SSYT for a vector-space of dimension $n = 3$:

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}$$

Then the 8 basis representants are

$$\begin{array}{rcl}
\widetilde{\varphi}_1(v_1, v_2, v_3) &=& e^1(v_1)e^1(v_3)e^2(v_2)\\
\widetilde{\varphi}_2(v_1, v_2, v_3) &=& e^1(v_1)e^1(v_3)e^3(v_2)\\
\widetilde{\varphi}_3(v_1, v_2, v_3) &=& e^1(v_1)e^2(v_3)e^2(v_2)\\
\widetilde{\varphi}_4(v_1, v_2, v_3) &=& e^1(v_1)e^2(v_3)e^3(v_2)\\
\widetilde{\varphi}_5(v_1, v_2, v_3) &=& e^1(v_1)e^3(v_3)e^2(v_2)\\
\widetilde{\varphi}_6(v_1, v_2, v_3) &=& e^1(v_1)e^3(v_3)e^3(v_2)\\
\widetilde{\varphi}_7(v_1, v_2, v_3) &=& e^2(v_1)e^2(v_3)e^3(v_2)\\
\widetilde{\varphi}_8(v_1, v_2, v_3) &=& e^2(v_1)e^3(v_3)e^3(v_2)
\end{array}$$

These basis functions representants are then symmetrized by the Young projector

$$\varphi_I = c_T \widetilde{\varphi}_I,$$

associated to the SYT T, i.e. first symmetrizing arguments 1 and 3, and then antisymmetrizing arguments 1 and 2:

$$\begin{aligned}
\widehat{\varphi}_I(v_1, v_2, v_3) &= (a(T)\widetilde{\varphi})(v_1, v_2, v_3) \\
&= \widetilde{\varphi}_I(v_1, v_2, v_3) + \widetilde{\varphi}_I(v_3, v_2, v_1)
\end{aligned}$$

and then

$$\begin{aligned}
\varphi_I(v_1, v_2, v_3) &= (b(T)\widehat{\varphi}_I)(v_1, v_2, v_3) \\
&= \widehat{\varphi}_I(v_1, v_2, v_3) - \widehat{\varphi}_I(v_2, v_1, v_3) \\
&= \widetilde{\varphi}_I(v_1, v_2, v_3) + \widetilde{\varphi}_I(v_3, v_2, v_1) \\
&\quad - \widetilde{\varphi}_I(v_2, v_1, v_3) - \widetilde{\varphi}_I(v_3, v_1, v_2)
\end{aligned}$$

for all $I \in \mathrm{SSYT}(\lambda)$

6 Differential operators

We want to understand differential operators acting on differential tensor forms.

Let

$$\sum_{I \in \mathrm{SSYT}(\lambda)} f_I(x)\varphi_I(v_1, \ldots, v_k)$$

It's derivative is the $k+1$ form

$$\sum_{I \in \mathrm{SSYT}(\lambda)} df_I(v_{k+1})\varphi_I(v_1, \ldots, v_k)$$

Howto split this into irreducible representations? Pieri map?

7 Non-constant coordinates

No idea where this will go ... Let $\phi : \widehat{U} \to U$ be a diffeomorphism between open sets in $\mathbb{R}^n$. A tensor must transform like

$$f(\phi(\hat{x}))(\phi' v_1, \ldots, \phi' v_k) = \hat{f}(\hat{x})(v_1, \ldots, v_k)$$

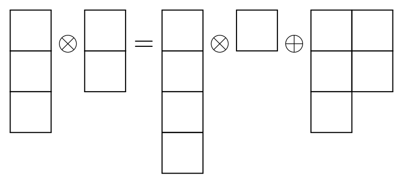
Taking derivatives on both sides into direction v_{k+1}

$$\begin{aligned}
& f'(\phi(\hat{x}))(\phi' v_1, \ldots, \phi' v_{k+1}) + f(\phi(\hat{x}))(\phi''(v_1, v_{k+1}), \phi' v_2, \ldots) + \ldots \\
= \;& \hat{f}'(v_1, \ldots v_{k+1})
\end{aligned}$$

Only if we alternate, terms with ϕ'' cancel out due to symmetry.

8 Moving a box

Does this make sense:



In other words

$$\Lambda^p \otimes \Lambda^q = \Lambda^{p+1} \otimes \Lambda^{q-1} \oplus SYT(p, q)$$

Dimension formulas for SSYT(n) match.

Can we compute the Young-projector onto SYT(p,q) by projecting onto (p,q) double forms, and subtracting the projector onto (p+1,q-1) double forms? Works at least for (p,q) = (2,1).

[thx AI for standard references]

References

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